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ORIGINAL ARTICLE

Optimal dispersion and central places

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Abstract This paper presents research into optimal dispersion models as applied to central places. The literature regarding location optimization and central places is reviewed and the motivation for employing dispersion models is identified. Models that employ the objective of maximal dispersion in the context of central places are formulated and solved in the context of both single- and multiple-good systems. Two methods for generating multiple-good systems are presented: a multiple-type dispersion model and a *K*-value constraint set formulation. Sequential solutions to dispersion models demonstrate how a system of central places could develop over time. The solutions to these models generate the patterns of central places expected under the organizing principles of central place theory. The objective of maximal dispersion is posited as both a motivating factor in central place location decisions, and as the optimal outcome of a mature system of central places.

Keywords Dispersion · Central place theory · Linear programming · Optimization

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1 Central place theoretic dispersion

Central place theory (CPT) has for over a half-century been considered to be one of the principal components of pure geographical theory (Berry and Garrison 1958). CPT provides a partial justification for an emphatic belief in the existence of a theoretical geography independent of any set of mother sciences (Bunge 1966). Geographic researchers have looked to CPT as an example of a search for fundamental spatial laws. Its elements are deductively linked, and its outcome is purely spatial. Among geographers there are those who assert that it is one of the most important and useful theoretical models in the study of geography (Tobler 1993).

The purpose of this article is to present the findings of research into optimal dispersion models as applied to central places. More specifically, models that employ the objective of maximal dispersion generate patterns of central places that are identical to the patterns expected under the organizing principles of classical CPT. This research does not attempt to either confirm or refute the social or economic processes that have previously been posited as underlying CPT. Rather, results are presented that demonstrate for the first time that under the classical assumptions of CPT an objective of maximal spatial dispersion will—by itself—generate the predicted central place patterns. Moreover, a recent multiple-facility dispersion formulation and the associated concept of repulsion measures are used to model hierarchical central place systems. A previously unformulated set of constraints are presented that force compliance with the *K*-value ratios between levels of the central place hierarchy, and the results generated from these constraints are contrasted with those from the repulsion-weighted method. These results suggest that dispersion, while most certainly not the only process underlying the location of urban centers, deserves attention as a motivation for location, and provides a mechanism through which central place patterns could develop over time.

2 Literature review

This is not the appropriate forum to exhaustively examine the enormous literature that surrounds CPT (See Berry and Pred (1965) for a bibliography only through 1964). Instead, this review examines 1) those elements of CPT that suggest that a dispersed pattern of central places is the theoretical outcome of a developed urban landscape, and 2) the use of optimization as applied to CPT.

2.1 The assumptions of central place theory

Walter Christaller outlined the theory of central places in his Ph.D. dissertation 1933 (Christaller 1966). The primary question Christaller asks is: ‘‘are there laws which determine the number, sizes, and distribution of towns?’’ He clearly believes that there are such ordering principles, and he accepted the notion that when searching for universal spatial laws, rather than descriptive idiosyncrasies, one may legitimately simplify the space to remove the particulars and to distill the generalities. His simplifying assumptions stated that towns develop on an isotropic plain with no variation in its physical character, its transportation facilities, or its productive capacity. A population with uniform characteristics was evenly dispersed on that plain. Those persons would purchase goods from the nearest supplier, and they would each have the same propensity to consume. Each good or service offered in a town had a range and threshold.

When considered simultaneously, Christaller's assumptions had the following consequences: The agricultural plain is eventually divided into non-competing, hexagonal (with sufficient competition), complementary regions, with a central place at the center of each. The largest central places offer all the goods that the population demands and can afford, requiring the co-location of facilities that sell different central goods, which determines the size of the complementary region. Lower-order centers are nested within the regions of higher-order centers. Christaller proceeded to show that these consequences of his assumptions are consistent under three organizational schemes. The marketing, traffic, and separation (administrative) principles provide arrangements of the central places in an area based on consumer behavior, traffic patterns, and human social and political organization, respectively.

August Lösch (1954) mathematically compared shapes of potential market areas (i.e., circles, squares, triangles, and hexagons) and showed that the honeycomb is the most advantageous shape for economic regions. Christaller's and Lösch's findings demonstrate that a hierarchical, dispersed pattern of central places and their associated market areas is the *optimal* arrangement under the simplifying assumptions given above. Subsequently, a body of CPT literature has developed that expands on this notion of optimality.

2.2 Central place theory and optimality

Much research into the optimal location of central facilities has been based on inferences of economic motivation from the assumptions of CPT as the basis for measuring optimal performance. In a review of alternative formulations (Beaumont 1987), the majority of the models specify an objective of maximizing demand. More complex models incorporate a notion of price elasticity of demand (Griffith 1986). Other studies have focused on minimizing costs: including transport and commuting costs between central places (Puryear 1975), and consumer shopping costs (Kohsaka 1984). Several franchise optimization models have been developed including a market share model which encourages avoidance of the locations chosen by a competing franchise, and a competition ignoring model which attempts to maximize market penetration by a single franchise with no concern for the locations of outside competitors (Goodchild 1984). The structures of these models are based on two seminal location–allocation problems, the p-median and p-center problems, respectively (Hakimi 1964). Another family of franchise-based central place optimization models addresses the need of a franchiser to expand the existing system of outlets (Ghosh and Craig 1991).

While the examples given above vary in the nature of the economic principles that are employed to develop an optimal central place system, they are consistent in their generation of only a single level of a central place hierarchy. The generation of multi-level (or multi-good) central place hierarchies has received substantially less attention from researchers and has been described as the "…major characteristic that must be addressed if location–allocation models are to be employed to operationalize central place concepts" (Beaumont 1987). One effort that focuses on this issue generates a hierarchical spatial system by minimizing the total cost incurred by both producers and consumers of a set of ordered goods, while serving the total demand (Dokmeci 1973). Another effort used both top–down and bottom–up approaches to determine spatially efficient multi-level systems with the p-median objective (Fisher and Rushton 1979), while a third sought to locate two levels of facilities using the same objective (Narula and Ogbu 1979). Still another multi-level model maximizes the number of firms that can coexist in a market (Kuby 1989). The rationale for this objective is based on the intuitive notion—attributed to Lösch—that additional firms will locate until excess profits are driven out of the system. By including the concept of inner and outer networks, and a set of constraints that enforce the symmetry of demand allocations, Kuby does, in fact, produce some solution patterns that are consistent with the three *K*-value systems (3,4,and 7) given by Christaller. In some cases additional nodes are added to the network in order to produce results consistent with theory. In essence, this model is validated by its production of results consistent with theory, although both modifications to the model that are not explicit representations of elements of the theory, and modifications to the network of nodes were required to achieve this validation. Another body of work shows that location optimization problems (including those dealing with hierarchical systems) can be solved with the use of Voronoi diagrams (Boots and Shiode 2003; Okabe and Suzuki 1997). Although these problems have not been cast directly in the context of CPT, the space partitioning nature of the Voronoi polygons and the fact that many of the objectives listed above have been implemented in this way suggests that such an application is reasonable.

Storbeck (1988) explored the arrangements of central place systems through a "natural" slack-covering model that locates central places on a triangular lattice in such a way that both market coverage of demand and market overlap are maximized. The range is included as a maximum service distance. In an extension to this work, Storbeck (1990) developed a protected threshold covering (PTC) model where each located central place maintained a protected inner range or threshold. The protected threshold model can be structured to mimic each *K*-valued central place system. In subsequent work a multi-objective model is formulated to explore the design of franchise outlet systems (Current and Storbeck 1994).

The range of optimal location objective functions that appears in the literature underscores the importance of choosing a function in which one can have faith in the validity of its underlying assumptions, and highlights the fact that many formulations address only one or two elements of a larger problem to be solved (Rushton 1987). The objective selected here is based on the observation that classical CPT requires a dispersed population as an input, and generates a hierarchical system that contains centers at each level that are maximally dispersed. These fundamental elements of the theory suggest that the formulation of a maximal dispersion model may be an appropriate means of generating a hierarchy of central places in order to test it against the expected outcome of the theory. If that test proves successful, a reconsideration of the impulses behind the geometric outcome may be appropriate.

3 Central place models of dispersion

Regardless of a particular researcher's interpretation of central place objectives, there exists a widely accepted, explicit, spatial form that can be used as a standard by which to judge any given central place model. This spatial form is the dispersed, nested, hierarchical system of hexagons described by Christaller and Lösch. The authors recognize that CPT is an example of process-form reasoning. That is, if a model (a) accurately reflects some process underlying the location of central places, and (b) correctly generates the classical CPT form, then this model allows the user to confidently explore selectively relaxed versions of the model and the consequences of those relaxations. If the *form* is generated, the *process* captured by the model may be of significant interest, and deserving of further examination. The purpose of this research is to determine if the process of dispersion can be considered a reasonable motivation for the location of central places based on the forms generated by that objective, and their correspondence with the well-defined geometric forms described by the classical theory.

3.1 Dispersion as form and process

In an effort to model the evenly dispersed population assumed in CPT as a set of discrete points, demand is commonly represented as a triangular lattice of points. This lattice of points also serves as the set of potential central place locations. This set of population points is not only evenly dispersed, but is, in fact, maximally dispersed according to the measure of the mean nearest neighbor distance (Ebdon 1988). Moreover, this population always seeks optimality, in that they always purchase a single central good from the nearest supplier (trip chaining or multiple-good trips could complicate this notion of optimality), thus minimizing the distance they need to travel and the concomitant transportation cost. These characteristics are reflected in the maximally dispersed set of optimal central places expected by theory. There is no question that dispersion is the form of both the inputs to and the outputs from CPT.

Given this well-accepted presence of dispersion in the form of CPT, this research is concerned with dispersion as a process. Can dispersion be an element of the purposive maximizing behavior, as defined by Eaton and Lipsey (1982) that is presumed to underlie the location of central places? Is the motivation to disperse part of the process that leads sequentially locating retailers to the eventual equilibrium state? The notion of sequential location of firms is significant (Prescott and Visscher 1977; West 1981). No one imagines that a set of sellers simultaneously locates at a state of equilibrium. Even if only two sellers are choosing locations, they certainly would not locate in such a way that they are separated by their threshold distance—allowing them to only just eke out a profit—even though this state will eventually occur as excess profits are claimed by entering competitors. They would certainly wish to locate further apart in order to capture all excess profit available in the system at that time. This idea is supported by the spatial consequences of competitive duopoly where an expressed or tacit understanding of the benefits of dispersion provides for a lower total social cost of purchasing goods (due to lower transportation costs) (Hotelling 1929), and larger total market (and sales) (Devletoglou 1965). Maximal dispersion as a location objective seems viable (d'Aspremont et al. 1979) as long as the tension between the seller and the consumer (who prefers accessibility and convenience) is not so great as to preclude trade. Under the assumptions of CPT this condition is met if the range is not exceeded, even when the threshold is.

Given these long-standing and intuitive notions regarding economic maximization, it is not unreasonable to imagine a supplier of some good wishing to locate at the maximum distance from a competing supplier. By being maximally dispersed from other merchants, one who locates early in the development of the central place system increases the possibility that the surrounding population will patronize his or her location. If there is any flexibility in the range of a good based on the lack of other suppliers, this objective also maximizes demand since the entire population within the sphere of influence of that supplier will choose to patronize that center. This will occur even if the threshold for profitability is exceeded, until additional suppliers choose to locate in the system. Moreover, by choosing to maximally disperse over the plain—rather than locating adjacent to another supplier of the same good at a distance of twice the threshold for that good—both suppliers will more likely be located at a higher order center as the central place hierarchy matures.

Thus the process of dispersion may be advantageous for those providing central goods. This research explores dispersion as an objective that represents a motivating process (through maximizing distance) and generates the optimal forms expected by the theory of central places. Initially, a set of single-good systems is generated on several different triangular lattices using a maximal dispersion objective. This is accomplished through a process of incrementally developing the central place system by adding additional centers. The outcomes of this process are explored in terms of the K-value patterns that can be observed among the various lattices. Next, two approaches are used to generate multiple-good central place systems. First, a multiple-type maximal dispersion formulation is presented and the notion of repulsion measures is described. Secondly, the K-value ratios developed in classical central place theory (and observed in the single-good case) are explored and the difficulties in using them as constraints on the location of centers are demonstrated.

3.2 Single-good systems of maximally dispersed central places

As an initial example, consider the set of eight graphics in Fig. 1a–h. Figure 1a shows 23 points maximally dispersed over the plain in a triangular lattice. The next seven graphics show the optimal solutions for the single-type maximal dispersion problem (Erkut 1990; Kuby 1987). A mathematical formulation of this problem is:

$$\text{Max } Z \tag{1}$$

$$\text{Subject to: } Z \leq d_{ij} + M(2 - x_i - x_j),\ 1 \leq i \leq j \leq n \tag{2}$$

$$\sum_{i=1}^{n} x_i = p \tag{3}$$

$$x_i = 0,\ 1, \quad i = 1,\ 2, \ldots,\ n \tag{4}$$

where
i, j = indices of potential central place location sites,
n = the number of potential central place location sites,
p = the number of central places to be located,
d_{ij} = the distance from potential location i to potential location j,
M = a very large number (at least larger than the largest distance between any two potential central place locations),
Z = an objective function value to be maximized, and
x_i = 1 if a central place is established at potential location i, 0 otherwise.

This problem is referred to in the literature as the p-dispersion problem and was first developed as an extension to the p-center problem (Shier 1977). An optimal solution to this problem will locate p central places such that the minimum separation distance between any pair of places is maximized (Erkut and Neuman 1989). Although this maximal dispersion formulation has appeared in the literature before, it has never been cast in the context of CPT, even though a dispersed population is the classic input, and a dispersed pattern of central places is the expected outcome. In this example, this problem has been solved for values of p ranging from 2 to 8. The optimal locations for central places are displayed with a black hexagon symbol. For each optimal solution the isotropic

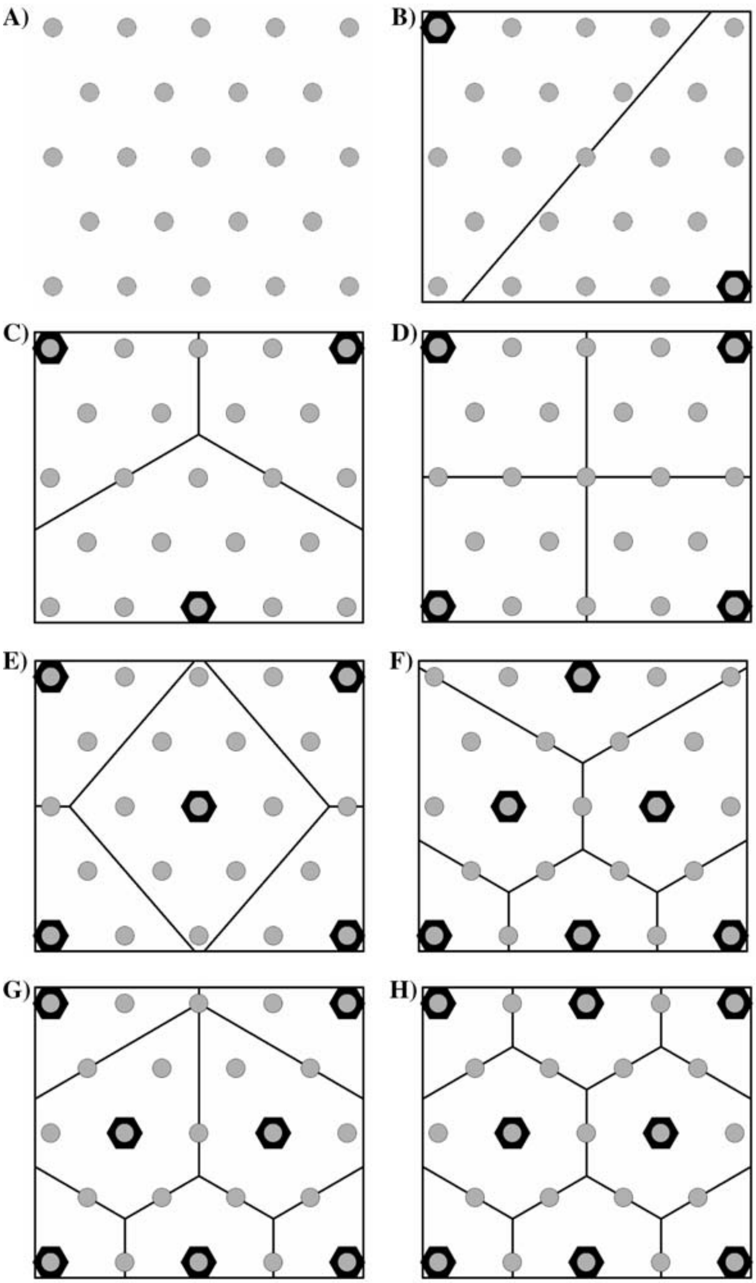

Fig. 1 Optimally dispersed single-type CPT solutions on 23 nodes

plain has been divided into Thiessen polygons showing the market areas of each central place.

3.2.1 Sequential growth

Figure 1 represents one progression of the sequential growth of a single-good central place urban system. The maximally dispersed solution for a single supplier is trivial; every point is equally optimal (although they are not equally optimal from a demand perspective, a perspective *not* under investigation in this research). If two competing suppliers are locating, however, the optimally dispersed solution is that shown in Fig. 1b, where two facilities locate at the extreme boundaries of the isotropic plain. The plain is divided into two equal, non-competing complementary regions. Since the population of the plain will always patronize the nearest supplier, the demand will be split equally between these two suppliers, and all of the demand will be served if the range of the good extends to the boundary of the Thiessen polygons. An initial arrangement where central places are located at the edge of the developing area is perfectly reasonable since, at this early stage of development these edge locations—which can be compared to coastal locations or the locations on the boundaries between developed and undeveloped regions—presumably benefit from contact with external systems.

In the absence of a constraint forcing central places to remain in their original locations, the addition of a third central place causes a relocation of a central place in order to achieve optimal dispersion. Figure 1c shows such an arrangement, where one of the central places from Fig. 1b has shifted to a new location, and a new central place has been added to the system. This process of relocation is not unexpected since urban centers are known to change their place in the urban hierarchy based on the location and importance of competing centers (Berry and Parr 1988). The shifting dominance of St Louis, MO and Chicago, IL during the development of the interior of the U.S. is one notable example (Cronon 1991). The graphic representation of this relocation could give the mistaken impression that there is a shift of an entire population from one location to another, particularly since we are only looking at one level of the central place hierarchy. This shift should instead be imagined as a relocation of some bundle of services from one location to another.

As the growth of the system proceeds, additional suppliers will be encouraged to locate on the plain as long as the threshold for profitability will allow it. With this sequential method, the process of locating additional centers (and sometimes shifting centers) based on the optimal solution for the maximal dispersion problem continues in this way until the isotropic plain is divided into non-competing, hexagonal regions that conform to the threshold constraint.

However, if—for a particular bundle of services—relocation is difficult (such as for services where capital intensive development is necessary for their provision), constraints can be added to fix facilities in place once they become part of an optimal solution. Doing so, however, can influence the arrangement

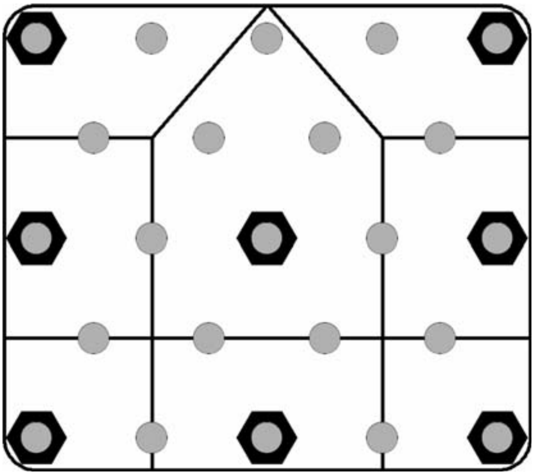

Fig. 2 Optimally dispersed solution with fixed facilities

of the mature system of central places and the value of the objective function. Figure 2 shows the maximal dispersion solution for eight facilities that have been incrementally added and fixed in place prior to another facility being added to the solution. This arrangement clearly differs from the result in Fig. 1h in that the expected central place pattern does not emerge, and the objective function value is inferior by 13.4%. This suggests that the location of fixed, or difficult to relocate, service facilities may be a fruitful avenue for future research into the discrepancies between ideal and actual central place patterns.

3.2.2 Edge and lattice configuration effects

The central place patterns generated through an objective of maximal dispersion are not immune to the edge effects that are present whenever the

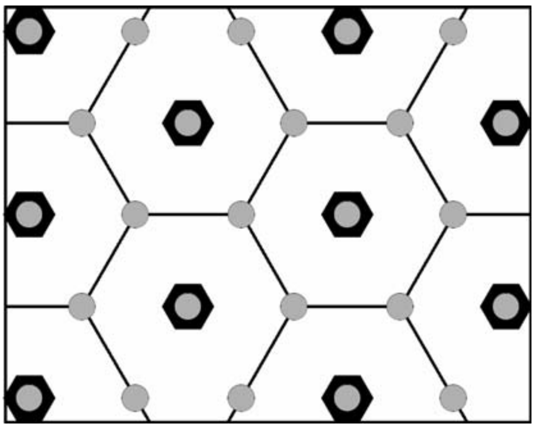

Fig. 3 Maximally dispersed arrangement on 25 nodes ($K = 3$)

assumption of an infinite, unbounded, continuous population is treated in the context of a discrete location problem (Kuby 1989). Such an effect can be seen in Fig. 3, which contains only one true market area that conforms to the $K = 3$ central place system, while other market areas are distorted by their proximity to the edge. Although additional rows or columns of demand points could be added to the lattice creating more true $K = 3$ market areas, this would simply push the edge effect further out without changing the result that a central place pattern is generated—in the interior of the lattice—through the implementation of the maximal dispersion objective. While methods have been developed to manage the edge effect (Kendall 1989), the applications of such methods in the context of maximal dispersion of central places are left for future research.

However, in this research the solution of maximal dispersion problems on triangular lattices with varying extents has exposed an effect related to edge effects that has not been noted in the literature. It can be demonstrated that the extent of the triangular lattice has an influence on the resulting K-value systems that appear during sequential generation of central place systems. Note that the optimal central place hierarchy in Fig. 1h conforms to the $K = 4$ arrangement (the transportation principle) described by Christaller. The $K = 3$ and $K = 7$ central place systems are never generated by the optimal solution to a maximal dispersion problem on this lattice. However, if the triangular lattice of population points and potential center locations is expanded to 25 points as shown in Fig. 3, and the number of centers is increased to 10, the optimal solution conforms to the $K = 3$ arrangement. The objective function has not changed, nor have the constraints, yet the nature of the central place system is different. Thus it appears that the nature of the mature central place solution is dependent on the number of lattice points and their configuration.

In order to explore this further, Fig. 4 presents a series of graphics showing the growth of a single good system on a triangular lattice of points based on the sequential solution of the maximal dispersion problem formulated above. The first few centers are located on the edge of the isotropic plain, as is consistent with the urban development of a sparsely populated region. When the number of centers reaches seven, a central place system matching the administrative principle ($K = 7$) is generated. When the number of centers increases to ten, a system matching the transportation principle ($K = 4$) is generated. When the number of centers further increases to twelve, a system based on the marketing principle ($K = 3$) is generated. At this point no additional centers can be located without locating immediately adjacent to an existing center, and the system can be considered to be fully mature, or at an equilibrium. This clearly demonstrates that any of the three classical central place systems can be generated solely through seeking the objective of maximal dispersion, on this particular triangular lattice.

It is as yet unclear what configurations will, in general, be associated with particular central place patterns. While the triangular lattice of points is well accepted as an abstraction of population or demand in central place research

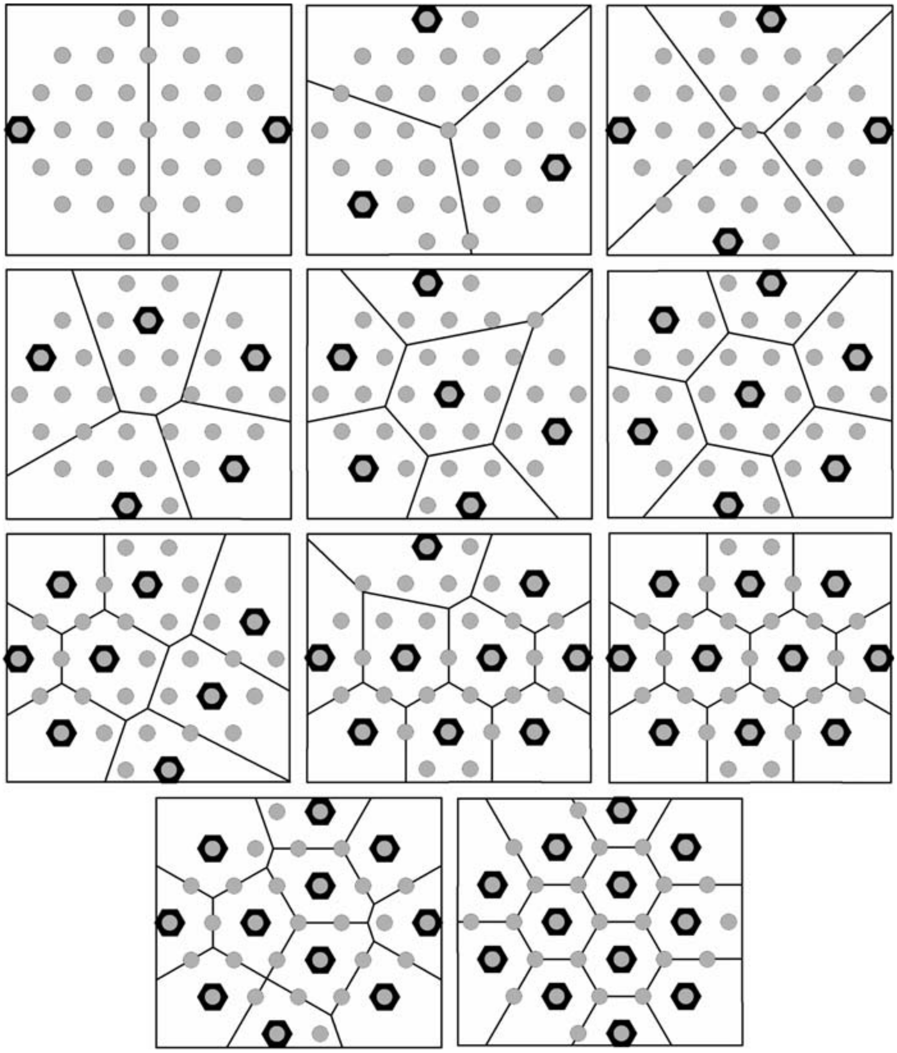

Fig. 4 Incremental single-good maximally dispersed solutions on a triangular lattice

(going back to Christaller himself) there is no consensus on what constitutes the "correct" extent of the lattice for research, there is no justification for the entire lattice to be configured as a square, rectangle, hexagon, circle, or any other particular shape, and there is no research into the nature what can be termed the configuration effects associated with triangular lattices of varying shape. While these additional research questions are of substantial interest to the authors, they are not the subject of this paper, and therefore further exploration of them is left for future research. Meanwhile the results that demonstrate that expected central place forms are generated by the optimal solutions to maximal dispersion problems are encouraging, as dispersion may then be a motivating factor in the development of central place systems. However, the single-good system does not represent a hierarchy of central places. The following section addresses this by presenting two methods of—and formulations for—generating multiple-good systems of central places by employing multiple-type dispersion models.

3.3 Multiple-good systems of central places

A multiple-good system of central places can be generated with the maximal dispersion objective in several ways. One potential solution procedure involves the solution of a series of problems, one for each good in the multiple-good system. The first problem finds the optimal solution for the lowest cost good in the same way that the central places in Fig. 3 were determined. These sites would then be the potential sites for the next level of central places. This approach is attractive in that it allows one to continue the progressive development of a system of central places that began with a single-good system described in the previous section. It is intuitive in that, over time, a center that sells a single good is more likely to expand in population and importance as a center, and will therefore become a likely candidate for the location of a supplier of a higher order good. Such a location is certainly a more likely candidate than a previously undeveloped population point, which would need to instantaneously become the location for the sale of multiple goods.

However, this approach presumes knowledge of what is historically considered an outcome or consequence of CPT. That is, it assures that complementary regions of lower-order centers will be nested within the complementary regions of higher-order centers, rather than allowing the structure of the model to determine the interactions between suppliers of the various types of goods. In order to accomplish this with a dispersion model for multiple-good systems of central places, the model must be made aware of the distinguishing characteristics of different types of central places and all supplier locations must be determined simultaneously. The following sections present two methods for doing so.

3.3.1 Multiple-type dispersion modeling for central places

A family of models for multiple-type discrete dispersion has recently appeared in the literature (Curtin 2002; Curtin and Church 2006). The various possible objectives for multiple-type dispersion models are related to those of traditional dispersion models (Erkut and Neuman 1990), but they differentiate between types of facilities that ought to be dispersed according to a set of repulsion measures. The more important it is to have facilities located away from each other, the stronger is their mutual repulsion value. This recent advance in dispersion modeling provides a new context for modeling hierarchical central place systems. A typical multiple-type dispersion formulation cast in the context of central places consists of:

$$\text{Max } Z \tag{5}$$

$$\text{Subject to: } Z \leq Q^{UV} d_{ij} + M(2 - x_i^U - x_j^V),\ i \text{ and } j = 1,\ 2,\ \ldots,\ n;\ i \neq j; \quad U \text{ and } V = 1,\ 2,\ \ldots,\ t; \tag{6}$$

$$\sum_{i=1}^{n} x_i^U = p^U,\ U = 1,\ 2, \ldots, t \tag{7}$$

$$x_i^U = 0 \text{ or } 1\ i = 1,\ 2,\ \ldots,\ n;\ U = 1,\ 2,\ \ldots,\ t \tag{8}$$

where the notation is the same as that defined for single-type dispersion problems above, with the additions of:

t = number of types of central places (levels in the hierarchy);
U, V = indices for central place types (levels);
x_i^U = 1 if a central place of type U is located at candidate site i, 0 otherwise;
Q^{UV} = measure of repulsion between central places of type U and of type V per unit distance of separation;
p^U = the number of central places of type U to be located

In this formulation constraints (6) force the value of the objective function Z to be less than or equal to the minimum of the repulsion weighted distances between any two central places of any type. A constraint exists for each tuple of potential locations and types of central place, with the exception that two central places of the same type cannot logically be located at the same location. If either (or both) of the two potential locations for a given constraint do not contain a central place of the type under consideration (that is, if x_i^U or x_j^V or both are equal to zero), then the objective function value Z need only be less than or equal to a very large number added to the repulsion weighted distance between the facilities. When both potential locations under consideration are assigned a central place of the types under consideration, the term containing the very large number (M) is equal to 0, and Z is constrained only by the repulsion-weighted distance between the central places. Since a constraint exists for all logical pairings of potential locations and types, Z must be less than or equal to the minimum repulsion-weighted distance between any two central places of any type. The sense of maximization in the objective function (5) ensures that a solution will be sought which maximizes this minimum repulsion-weighted distance.

Constraints (7) control the number of different types of central places to be located. As in the example of sequential central place development given in Fig. 1, the values of p^U could be varied to demonstrate how centers of low order goods develop first, and then higher order centers locate among these lower order centers. Constraints (8) require that all decision variables are equal to either 0 or 1, thus guaranteeing an integer solution.

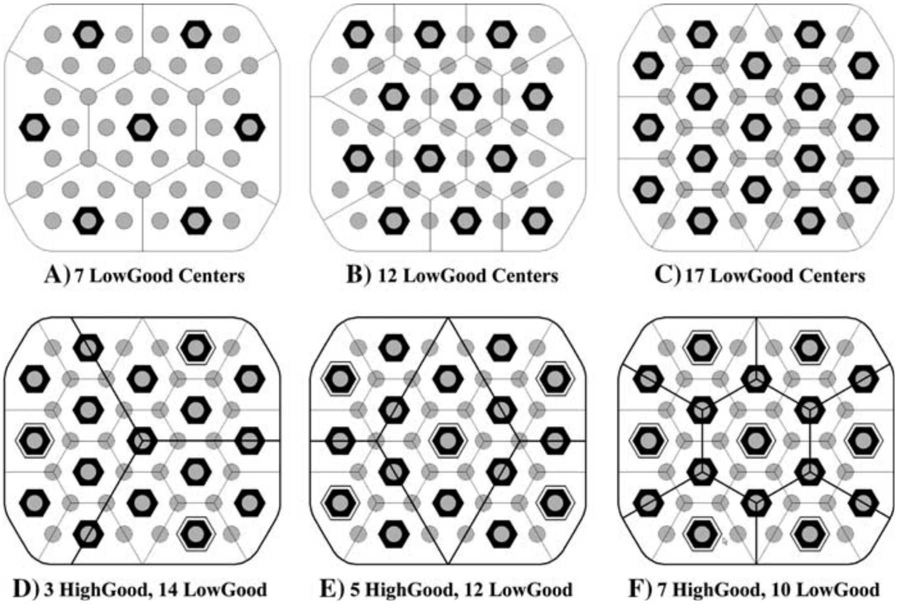

Fig. 5 Incremental multiple-type (2 level) optimally dispersed solutions

It is the repulsion measures in constraints (6) that encourage central places at different levels of the urban hierarchy to disperse. As an example, consider the triangular lattice of 45 population and potential central place location points, and the incremental optimally dispersed solutions for a two-good central place system given in Fig. 5. In this example there are two types of central places—those that sell only the lowest order good (here called a LowGood center) and those that sell the lowest order good and one other higher order good (called a HighGood center). There must be repulsion measures between pairs of LowGood centers and between pairs of HighGood centers. Since LowGood centers are nested within HighGood centers, the repulsion measures between them are equal to those between pairs of LowGood centers.

The repulsion measures are directly analogous to the threshold values for the goods sold at each type of center: a larger threshold is associated with a stronger repulsion value. Higher order goods are more expensive than lower order goods, and therefore require a larger threshold to ensure that the continued sale of the good is profitable. Additionally, people are generally willing to travel greater distances to purchase higher order goods, since they do so less frequently and are committing a greater proportion of their resources to the purchase. Therefore, both the threshold and the range values for higher order goods are greater. Given this, in order for a multiple-good system of central places to be generated, pairs of centers that sell higher order goods (in addition to lower order goods) must have a repulsion measure that is stronger than the repulsion measure for pairs of centers that sell only lower order goods. This will result in higher order centers locating further apart, with a larger market area surrounding them. In the example given here the repulsion measure for pairs of HighGood centers must be stronger than the repulsion measure for pairs of LowGood centers. In the example shown in Fig. 5 a repulsion value of 1 was applied to pairs of LowGood centers and a repulsion value of 0.34 was applied to pairs of HighGood centers. Although somewhat unintuitive, when repulsion values range from 0 to 1, a smaller repulsion value is stronger since it will be multiplied by distances in constraints (6), distances which the model seeks to maximize.

Using the multiple-type dispersion model with repulsion measures it is possible to either determine immediately the optimal arrangement for a mature system of central places using values for p^U that fill the triangular lattice, or to incrementally determine the growth of the urban system over time. If this second approach is desirable, one can first solve a series of problems for a single-good system as shown in Fig. 5 (only 6 of the 17 increments are shown). In our example the value of p^{lowgood} increments from 2 to 17 while the value of p^{highgood} remains at zero. When seventeen LowGood centers are located the single-good system is mature. At that point, a problem can be solved for a value of $p^{\text{lowgood}} = 15$ and a value of $p^{\text{highgood}} = 2$. In each successive problem the value of p^{lowgood} would decrease by 1 and the value of p^{highgood} would increase by one, essentially replacing a LowGood center with a HighGood center. When the value of p^{highgood} reaches 7 the multiple-good central place system is mature, and no additional centers of either type can be added without locating a center immediately adjacent to another center, violating a threshold constraint. Once again, $K = 3$ and $K = 4$ central place patterns are generated during the incremental location of facilities, solely based on optimal dispersion. Note that nesting of central places occurs (Fig. 5) even in the absence of formal constraints requiring it.

3.3.2 Multiple-type dispersion with K value ratios

Another method for generating multiple-good systems of central places involves the use of the *K*-value outcomes of central place systems developed by Christaller (1966). As mentioned in the discussion of single-good central places, there are three primary *K*-value systems in classical central place theory, although many more can and have been generated, including fractional *K*-value systems (Church and Bell 1990). The three primary *K*-value systems are $K = 3$, $K = 4$, and $K = 7$ which correspond to the marketing, transportation, and administrative principles, respectively. The fact that all of these central place systems appear at various times in the development of a single-good system of central objective suggests that a system could be constrained to conform to a particular *K*-value.

For each of the *K*-value systems, the corresponding value represents the total number of whole and fractional centers that are within the sphere of influence of a center at the next highest level of the central place hierarchy.

For example, in a $K = 3$ system each HighGood center market area contains a total of three whole or fractional LowGood market areas. In order to model mathematically this relationship among centers in the urban hierarchy, the expanded p^U constraints from the previous formulation (constraints 7) can be complemented by a series of constraints that control the number of each type of central place to be located based on the K-value ratio. For example, in a central place system governed by the $K = 3$ market principle, the following constraint set would maintain the appropriate values of p^U for each of the multiple types of centers to be located:

$$p^U = 2p^{U+1},\ U = 1, 2, ..., t - 1 \tag{9}$$

since a center of the next lower order is implicitly nested at the location of a higher-order center, the multiplier representing the K-value must be reduced by one in order to conform to the appropriate ratio. In the general case, if the variable R represents the K-value ratio multiplier, the value of p^U is given by:

$$p^U = (R - 1)p^{U+1},\ U = 1, 2, ..., t - 1. \tag{10}$$

A constraint would then exist for each of the multiple types of centers being located with the exception of the highest order center in the hierarchy (where $U = t$). In this last case, the value of p^{U+1} must be equal to zero since this is the highest-level center in the hierarchy. If a constraint of this type existed for $U = t$, this would force all values of p^U to be equal to zero.

Although these constraints control the relative number of central places at various levels in the hierarchy, the user must supply some initial information in order to seed the p^U computations. If the user specifies only a global value for p, then another set of constraints must be included to ensure that this global value is met. Either the original constraint from the general single-type maximal dispersion formulation that ensures that exactly p centers will be located (constraint 3) must be included in the formulation, or a constraint must be included that ensures that the individual p^U values sum to this global value:

$$\sum_{U=1}^{t} p^U = p \tag{11}$$

additionally, either the user or a constraint in the model must specify at least one of the p^U values. Alternatively, the model could potentially be designed to always assign a value of 1 to the p^U value for the highest-level center in the hierarchy based on the number of types of facilities to locate, and to then locate the appropriate numbers of facilities of each lower-order type. In summary, these constraints preserve the K-value ratios between centers at different levels of the central place hierarchy. The model itself determines the appropriate split between facility types based on the central place ratio.

On closer inspection, however, there are two problems with using the constraints on p as presented above. First, it is possible for the user to specify a global value for p with which it is impossible to generate values of p^U that satisfy the constraint sets. As an example, if a two-good central place system is being generated with the marketing principle, and the user chooses a global p value of 5, there is no way to logically satisfy the constraint sets. If $p^1 = 1$, p^2 must be equal to 2 for a global p value of 3. If $p^1 = 2$, p^2 must be equal to 4 for a global p value of 6. No integer values of p^1 exist for which the global p value will be equal to 5. In order to circumvent this problem, the user could specify only the p^U value for the highest order centers, and the number of lower order centers would be computed from this value and the K-value multiplier. As long as the number of potential facility sites is greater than the resulting global p value, a feasible solution could be generated.

A more challenging problem with using K-values to generate multiple-good central place hierarchies is that K-value relationships do not always hold. In the case of a fully developed central place system the K-value relationships will hold only in the central regions of the isotropic plain. When higher order centers are located near the boundary of the evenly populated region there may not be sufficient potential facility sites available to contribute to the market area of the center, and therefore the K-value ratio cannot be met for these centers. This edge effect can be seen upon inspection of the example multiple-good central place system in Fig. 5. At equilibrium, it can be seen that the market principle ratio of 3:1 holds only for the HighGood center located at the center of the region. The six HighGood centers that have been located along the edges of the triangular lattice, have only 2 1/6 or 2 2/3 centers of LowGood associated with them. The missing LowGood centers would be located outside the bounding edge of the triangular lattice. It is possible that the concept of inner and outer networks implemented by Kuby (1989) could be employed to overcome this difficulty, and the authors recognize that this is an empirical modeling problem rather than a theoretical one. Moreover, the K-value ratio method allows one to examine only the equilibrium state of a central place system. This method does not provide the ability to model the sequential growth of a system through the addition of newly locating central places that was successfully demonstrated with the maximal dispersion formulation.

4 Discussion, conclusions, and future research

In this paper optimal dispersion models have been associated with central places. First, a set of single-good central place systems was developed on a series of triangular lattices using a maximal dispersion objective. The hierarchical systems were seen to grow and mature through successive incremental increases in the number of dispersed central places. A range of K-value central place systems was observed as the outcomes of these incremental processes. An examination of how the extent and configuration of the underlying lattice of points influences the K-value central place systems that appear under the objective of maximal dispersion remains as a subject for future research.

Two systems were then presented to demonstrate the use of dispersion modeling in multiple-good central place systems. The first employed multiple-type dispersion models with a previously unpublished set of constraints on the value of p (the number of centers to locate) for each type of center. This model also included the use of repulsion measures to insure variation in distances among the various types of centers to be located. Since the concept of repulsion measures has only recently been added to the location science literature, the topic may well benefit from additional research. A second system for generating multiple-good central place systems was developed based on the K-value ratios first observed by Christaller. Although optimal central place hierarchies were developed through the use of K-value constraints, there were difficulties with edge effects, and the ability to demonstrate system growth was lost.

Most importantly, this research demonstrates that central place hierarchies can be developed solely with the objective of optimal dispersion. Thus, maximizing distances among center locations is viable as a means through which the growth of urban hierarchies can be examined. A dispersive objective appears to be advantageous to the seller of central goods in that the greatest possible market area is secured. Given this demonstration of a purely geographic motivation, perhaps dispersion can now be combined with some of the economic motivations developed earlier to obtain a more comprehensive understanding of CPT. A test of the outcome of optimal dispersion against a real world, temporal central place location pattern is also a subject for a subsequent paper.

Although not presented in the present work, the authors have also developed a formulation of an r-separation model related to central place location. This model seeks to locate as many facilities as possible in an area with a minimum level of dispersion (or separation). The r-separation constraint provides an additional method for modeling the concept of threshold. Such a model may also be more appropriate for franchise systems where the minimum sustainable market area is known in advance, and the siting of as many franchisees as possible is desirable. Since r-separation models are not explicitly maximizing dispersion, we leave that model and its variants as a subject for future research.

Certainly the impetus for humans to organize themselves into urban centers is the result of a complex interaction of spatial, economic, social, and behavioral factors. This research strongly suggests that maximal dispersion can play a role as a fundamental spatial motivation.

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