Operations Research

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Glossary

analytical hierarchy process A method for building a hierarchy of decision elements and comparing and ranking those elements.

branch and bound A search procedure in which relaxations of an integer linear program are organized in a tree structure and solved in order to bound the solution space of the root problem and converge on the optimal solution.

combinatorial complexity The measure of the size of or difficulty in solving a problem based on the number of possible combinations of decision variable values within the problem.

decision variables The elements of an operations research model that represent choices among possible alternative assignments of resources.

Delphi process A structured method for developing consensus among decision makers and experts regarding the significant factors to be included in a model.

enumeration procedure A method for examining all possible alternative solutions in order to identify an optimal solution.

heuristic solution method A "rule-of-thumb" algorithm that will rapidly produce high-quality (though not guaranteed optimal) solutions to a difficult problem.

interior point solution procedure An iterative procedure that identifies a solution within the boundary of the feasible region for a problem and improves on that solution until an optimal solution is found.

linear programming A method for modeling a complex problem as a set of linear functions, including an objective function and a set of constraints.

mathematical modeling The process of structuring complex systems as a set of mathematical functions.

objective function The mathematical expression of the goal of a complex system.

optimization An act, process, or methodology of making something (as a design, system, or decision) as fully perfect, functional, or effective as possible; specifically the mathematical procedures (as finding the maximum of a function) involved in this process.

resource constraint A mathematical representation of a limitation on reaching the objective for an optimization problem.

simplex method An iterative procedure for solving a system of linear equations given in a standardized form, representing constraints on a system.

Operations research is defined as the application of advanced analytical techniques in order to solve complex problems. Its dominant characteristic is the use of mathematical models to analyze problems. With origins in military strategic planning, operations research has found applicability in a wide range of industrial, commercial, and social contexts. Although many of the problems studied are highly combinatorially complex, advanced optimal and heuristic solution procedures have been developed to find alternative solutions for decision-making processes.

Operations Research Defined

The field of operations research (alternately termed operational research or management science) is defined as the application of advanced analytical techniques in order to solve complex problems. Operations research (OR) contains a set of tools used by those who must make organizational decisions. Often these problems involve the allocation of scarce resources in such a way as to achieve a goal maximally (such as profit or level of service) or to minimize some negative consequence of the operation of an organization (such as cost or environmental degradation). In order to solve a complex problem posed by the operation of an organization or system, this problem must be formulated in such a way that it can be efficiently analyzed.

The dominant characteristic of OR is that it constructs and uses mathematical representations, i.e., models, to analyze problems. These models commonly take the form of an objective function that defines the goal of the organization (or one of many goals), and a set of constraints representing the conditions within which the system must operate. Once the general version of problem is formulated, individual instances of that problem may be solved optimally in order to suggest specific allocations of the organization's resources. This solution will be the one that best satisfies the objective that is to be optimized. This distinctive approach is an adaptation of the scientific method used by other sciences.

History of Operations Research

The discipline of OR has its origins in the application of problem-solving techniques in a military context during World War II, when the practice was termed "military OR." In this context, the complex system was designed to wage war, and the components of that system were the enormous resources and requirements of military organizations. Examples of how OR was applied include the determination of the optimal fuse length for depth charges designed to combat submarines, the optimal deployment of radar stations and mining operations, and the optimal arrangement for convoys of ships.

Following the end of the war, two fundamental changes occurred in the practice of OR. First, the methods employed in the conduct of war were quickly adapted to a wide range of industrial applications. Rather than maximizing the efficiency of weapons systems, problems were formulated to increase the efficiency of manufacturing and transportation systems. Second, the development of the simplex method by G. B. Dantzig in 1947 revolutionized the practice of solving mathematical models. The simplex method is a procedure for solving a system of linear equations given in a standardized form, representing the constraints on a system. This iterative method can efficiently evaluate very large numbers of such constraints and associated variables. Due to the computational complexity of such problems and the potential for astronomically large numbers of possible solutions to evaluate, the simplex method effectively unlocked the solutions to a vast number of problems that could not otherwise have been solved.

The two changes in the practice of OR have brought about the steady growth in the development and application of OR techniques for over 50 years. Moreover, these techniques have demonstrated that the efficiencies gained by mathematically modeling a system and solving that model optimally are significant and can in some cases be extraordinary. Recent findings show that major corporations have realized savings in the hundreds of millions of dollars attributable to the implementation of optimization techniques.

An Approach for Conducting Operations Research

Although the mathematical model provides both the message for decision makers and the structure for conveying that message, the model alone is only one part of a process for conducting OR. There are commonly five stages in this process.

Stage 1: Understand and State the Problem

As in any research project, the first step is to state the problem clearly. In the context of OR, such a statement involves an understanding of the system to be modeled and the environment within which the system operates. Understanding the system requires that there is a known goal (or set of goals). Examples of goals are to maximize profit in the context of a commercial enterprise, to minimize cost for a particular manufacturing process, or to minimize distance traveled in the case of transportation applications. Moreover, there may be many factors that influence the extent to which the goal can be met. In terms of commercial gain, there are limits on costs of acquiring goods and constraints imposed by both markets and regulations. In terms of manufacturing processes, there are set requirements for inputs and scheduling considerations. Transportation functions depend on the origin and destination locations, the locations of stops, mode choices, and a host of other factors. All of the pertinent factors must be determined in order to have a reasonable understanding of the problem. These factors can be determined through consultation with the managers of the system and others who are experts in the field, perhaps through a Delphi process. It may be appropriate to determine the importance of different factors through the use of an analytical hierarchy process or some other method of ranking these factors. With an understanding of the factors that influence the system, a clear statement of the problem can be generated.

Stage 2: Formulate the Problem Mathematically

In any research project, models are built in order to find a useful, albeit simplified, representation of a real-world situation. It is no different in OR. A mathematical model is generated from the understanding of the problem gained in stage 1 of the research process. The model often consists of a single objective function that reflects a simplified vision of the goal to be met. The objective function also serves as the quantitative performance measure for the system being modeled. A series of mathematical constraint functions represent simplified versions of the limitations that must be met in the system.

Within these functions many different decisions must be made in order to evaluate the system. Examples of decisions include which roads to travel in order to minimize distance, which locations to choose for warehouses, or how many units of manufactured goods to ship from a particular warehouse to a retail store. These decisions are represented within the functions in the mathematical model with decision variables, and it is the value of these variables that must be determined in such a way as to optimize the system. Decision variables may be binary (e.g., either a road is traveled or it is not), integer (e.g., only whole units of manufactured goods may be shipped), or fractional (e.g., any number of gallons or fractions of gallons of water may be pumped in order to meet the needs of a community). Constants or weights may be associated with particular decision variables if relevant and accurate data exist.

Consider as an example the mathematical formulation for a common OR problem known as the "knapsack" problem. This problem arises when there is limited space to carry or include items (such as in a knapsack) and the objective is to select those items that will be most valuable for inclusion in the limited space. This could pertain to the loading of products for delivery into trucks with limited space or limited weight-carrying capacity. First, the notation used in the formulation is defined as follows: j is the index of item types, N is the number of types of items, c_j is the value of item type j (where $j = 1, 2, \dots, N$), a_j is the weight of item type j (where $j = 1, 2, \dots, N$), b is the limit on the total weight of the items that can be included, and x_j is the number of items of type j that are included in the knapsack. The objective or goal is to maximize the value of the items included in the backpack, and thus the objective function can be written as

$$\text{Maximize } Z = c_1x_1 + c_2x_2 + \cdots + c_Nx_N,$$

or alternatively as

$$\text{Maximize } Z = \sum_{j=1}^{N} c_jx_j.$$

This objective function is subject to a constraint on the total amount of weight allowed (room in the knapsack or truck). This constraint can be written mathematically as

$$a_1x_1 + a_2x_2 + \cdots + a_Nx_N \leq b,$$

or alternatively as

$$\sum_{j=1}^{N} a_jx_j \leq b.$$

The x_j variables are the decision variables for this formulation. It must be decided how many of each type of item (if any) ought to be included in the knapsack in order to maximize the objective function, while respecting the constraint. In practice, it is impossible to put negative numbers of items in a knapsack, thus a set of constraints is usually generated to ensure that the decision variables hold only non-negative numbers of items. These can be written as $x_1 \geq 0, x_2 \geq 0, \dots, x_N \geq 0$, or alternatively as $x_j \geq 0$ for all $j = 1, 2, \dots, N$. Depending on the nature of the items to be included in the knapsack, these decision variables may be further constrained. For example, if the items to be placed in the knapsack cannot be broken into smaller pieces, they must be constrained to be only integer values. Moreover, if only one item of each type may be included in the knapsack, then the decision variables are binary and $x_j = 1$ if an item of type j is included in the knapsack, and $x_j = 0$ otherwise.

Although the knapsack problem is only one of many different types of problems that are commonly solved in OR, the mathematical formulation is typical of many different problems: there is an objective function, a constraint on the available resources for the system, and a set of constraints on the values of the decision variables. This common type of formulation is called a linear programming formulation, and such formulations are perhaps the chief research area in OR. However, several other research areas command attention, including critical path analysis, dynamic programming, goal programming, nonlinear programming, decision analysis, game theory, Monte Carlo simulation, and queuing theory. Due to the limited space here, the focus remains on using linear programming to model complex systems.

Stage 3: Solve Instances of the Problem

Once an appropriate model formulation has been developed, it must be used to solve real-world instances of the problem. Although there is a fairly large toolbox of solution procedures and variants of such procedures, they can be grouped into five major categories: graphical solution procedures, enumeration methods, simplex-type solution methods, interior point methods, and heuristic solution methods.

Graphical solution methods depend on the fact that all of the parts of a typical mathematical formulation are linear functions. Those functions can therefore be graphed in Cartesian coordinate space in order to determine the optimal solution. Consider a problem instance with two decision variables (x_1 and x_2) and with two resource constraints ($3x_1 + 5x_2 \leq 20$ and $4x_1 + x_2 \leq 15$). The decision variables must be non-negative. The objective function is to Maximize $Z = 2x_1 + x_2$. These functions can be graphed as in Fig. 1. Each axis in the coordinate system represents one of the two decision variables. Each of the constraints can be graphed as linear equalities, whereby one side of the line represents feasible values for the decision variables and the other represents values that would violate the constraint. Taken together, all of the constraints (including the non-negativity constraints) define the feasible region for the problem instance. This area is shaded in Fig. 1. The objective function cuts through the feasible region, and its exact placement depends on the values of the decision variables. With the graphical method, it is easy to see that the objection function value will be greatest when the decision variables have values that allow it to be drawn through the intersection of the two resource constraint lines. Although the graphical method for solving OR problems with linear functions is a very intuitive tool, is usefulness is limited to those cases in which there are only two or three decision variables, since it is difficult to represent four or more dimensions graphically in such a way that they can be easily interpreted. Therefore, other solution procedures must be employed for more complex problem instances.

When there are small numbers of constraints and decision variables in the problem instance, it is conceivable that a complete enumeration procedure could be used to find the optimal solution. That is, each possible combination of decision variable values is tested and the associated objective function value is found. One or more of those combinations will be optimal. It may be more efficient, however, to employ a search method such as "branch and bound" to eliminate some nonoptimal solutions from consideration. The branch and bound method divides the feasible region into partitions in a tree structure and uses the solutions to subproblems to bound the objective function and "prune" branches of the solution tree. In practice, however, many of the problem formulations that must be solved in OR belong to a large family of problems that have been proven to be "NP-complete" (where NP refers to nondeterministic polynomial time). Simply put, this means that there are no algorithmic solutions to such problems, and the size

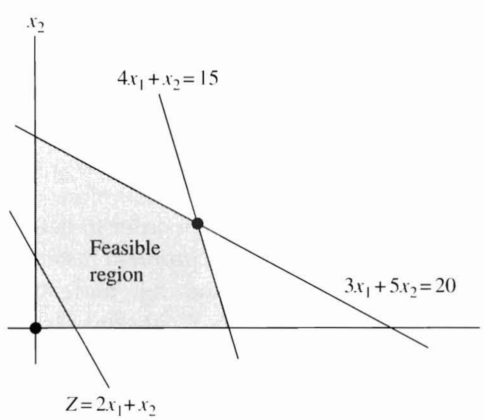

Figure 1 Graphical solution to a two-dimensional problem.

of the problems (as measured by the number of decision variables and constraints) may grow exponentially. Due to the massive resources (in terms of computing time, memory, or storage) that such problems require in order to find a guaranteed optimal solution, it is effectively impossible to solve them through any enumeration procedure. Therefore, an appropriate solution procedure for a problem instance must be chosen based on an understanding of the combinatorial complexity of the problem.

To demonstrate the notion of combinatorial complexity in the context of common OR problems, consider the simple system present in a game of straight pool (billiards). There are 15 billiard balls numbered sequentially, and these must be arranged (in any order) in a rack with 15 places. In the general case, the number of alternatives will be as follows:

$$\text{Number of alternatives} = \binom{n}{p} = \frac{n!}{p!(n-p)!}$$

where n is the number of locations in the rack and p is the number of billiard balls to locate among those potential locations. In the special case of a billiard rack, n is equal to p, so there are 15! (15 factorial) possibilities, or 1,307,674,368,000 possible arrangements of the balls in the rack. In order to determine the optimal solution that describes the best way to arrange the balls through simple enumeration, the objective function must be evaluated for each of the possible arrangements. Depending on the complexity of the objective function chosen for the rack, and on the computing power available, this problem may be unsolvable through inspection. Even a solution protocol that can evaluate the objective function 10,000 times per second would need to run for over 4 years in order to guarantee that the optimal solution had been found. Recall that this problem represents a system with only 15 components! The problems of managing a complex industrial application dwarf this problem by comparison. Given such combinatorial complexity of even small instances of problems in OR, it is clear that enumerative solution procedures are insufficient for many applications.

The development of the simplex method revolutionized OR by providing a standard technique for solving even large optimization problems of the form just described. It is an iterative method that explores the boundary of the feasible region, improving the objective function value with each iteration until the optimal solution has been found. Often the simplex method is used in conjunction with a branch and bound procedure to obtain integer optimal solutions. In other cases, additional constraints (termed "cutting planes") are included in the formulation in order to eliminate parts of the feasible region containing fractional solutions. The "transportation" problem, the "assignment" problem, and the "transshipment" problem are examples of the common problems pertinent to the simplex method. Special versions of the simplex method have been developed to more efficiently solve some problems with particular structural characteristics. The "transportation," "assignment," and "transshipment" problems are among these. These special case solution procedures are sometimes termed "network flow" solution methods.

Interior point solution methods are an alternative to the family of simplex solution procedures. These are iterative methods that find feasible solutions inside the boundary of the feasible region (rather than on the boundary), and at each iteration they find a solution that provides a better objective function value, until the optimal solution is found on the boundary.

When the problem size proves too large for any of the enumeration or iterative methods, heuristic solution methods may be the only practical means of finding a solution. Many heuristics (or rules of thumb) have been developed and tested on problems for which the optimal solutions are known. It is presumed that if a heuristic works well on small problems, it will likely work well on larger instances of the same problem. Some of the most common heuristics include a family of interchange heuristics for which an initial solution is chosen and successive interchanges of the decision variable values are used to search for solutions that improve the objective function value. Interchange heuristics are susceptible to becoming trapped at local optima that may be far from the global optimal solution for the problem, and Tabu (as in "taboo," referring to prohibition) versions of these heuristics have been developed to overcome this limitation. Simulated annealing heuristics use an analogy to the process of heating and cooling metal to harden it, in order to move toward the optimal solution. Lagrangian relaxation heuristics are used to find upper bounds on a problem, ideally narrowing the solution space such that the optimal solution is determined. Although heuristic solution procedures may be the only reasonable method for determining solutions to large problems, and although they may find the optimal solution, there is no guarantee that they will do so.

Stage 4: Validate and Interpret the Results of the Model

Once a general formulation for the model has been formulated and instances of the problem have been solved optimally or heuristically, the researcher must examine the solution and reexamine the model that generated it. Only through the processes of validation and interpretation of the solutions that are generated can the researcher determine whether the model accurately represents the problem environment. The quality of the optimal solution is a function of the suitability of the objective function, the constraints, and the parameters. The nature of mathematical models guarantees that any objective function will be limited in its precision and open to interpretation.

There are several flaws from which models commonly suffer, and which ought to be considered in the validation process: extraneous variables, missing variables, misidentified parameter values, and structural problems. It is possible to use standard statistical analyses such as correlation, regression, and analysis of variance to test for the significance of the variables to the solution. If a variable is not significant to the solution, it may be appropriate to eliminate it and perhaps make the problem easier to solve. Conversely, measures of explained variance might suggest that relevant variables are missing, although they will not help to determine which ones. Sensitivity analysis is useful in determining the amount by which the parameters can be altered before the generated solution will no longer be optimal. The parameters that are varied can include economic and social characteristics such as prices, demands, or population figures, and positional variations in site locations or distance measurements in the case of location optimization problems. If the solution is very sensitive to changes in a particular parameter, then those values must be determined with greater care.

If the model appears to be valid, a critical interpretation of the model should be conducted in the light of the model results. Simply visualizing the solution or presenting it to decision makers can bring to light structural flaws in the model. The optimal solution may expose a missing constraint that would preclude the solution from ever being implemented. The interpretation must be conducted with awareness that the optimality of the solution is always relative to a carefully stated set of constraints, and these constraints are always surrogates for and simplifications of reality. Satisficing solutions that consider qualitative or subjective elements may be close to optimal, but more appropriate in terms of constraints that are well known but not easily expressed as a mathematical function.

Although validation and implementation are discussed here as a fifth step in a research methodology, elements of these processes may be conducted throughout the research program. This highlights the importance in stage 1 of the identification of decision makers and experts for determining objectives, constraints, and significant parameter values.

Stage 5: Implement the Model—Use the model to Explain, Predict, and Decide

The final stage in the OR process is the implementation of the model solutions. Implementation of the solution is the only test of the validity of the results of the research. Because the OR process is concerned largely with optimality, it is tempting to assume that after careful formulation, solution, and validation there could be no possible reason to reject or delay implementation of that optimal solution. In practice, the generation of a valid optimal solution may be less than half the battle. The implementation of a new solution for a complex system may cause massive change in many different but related systems. Resistance to these changes may be substantial and can prevent the implementation of the optimal solution.

In order to ensure implementation of the results of the OR process, those in a position to authorize its implementation must believe that the model is a reasonable simplification of the system and its environment, and they must understand the structural assumptions built into the model. Such understanding can be gained only through clear communication of the model and the ways in which it was generated, and through the involvement of all interested parties. Because OR tries to improve the efficiency of the system under consideration, it implies a tacit criticism of the existing arrangement. Therefore, in order to avoid resentment among those who manage the current system, the results of the research must be considered to be an imaginative exercise in tactful communication and persuasion. Toward this end, decision makers must be presented with a range of alternative solutions, so that they can select the alternative that can be implemented at a reasonable cost. Although they may risk the loss of optimality, they will be able to weigh the cost and time of implementation versus the potential gains.

If the solutions are implemented, it must also be considered that the system is very likely not static, and it will need to be controlled and monitored over its useful life. Changes in the environment may invalidate the assumptions of the model. Additional constraints may be identified, parameter values may change substantially, or objectives may even change. The application of OR requires that all involved are prepared for changes that may have long-lasting implications for the all of the related systems.

Applications of Operations Research

In the few decades since its origin as a tool for military strategy, OR has become involved in an astounding number of applications areas. Due to the interdependence between the military and industrial activities, there was a quick acceptance of this body of methods in manufacturing. OR quickly proved to identify efficiencies in various production systems, including assembly processes, the blending of ingredients, inventory controls, optimal product mixes, trim loss applications, and job shop scheduling. Later, high-technology manufacturing was to use OR to assist in circuit layout design and multiprocessor assignment applications.

Managers of commercial interests other than manufacturing also welcome the cost-effective organizational changes that OR can suggest. Personnel directors can more efficiently schedule a workforce, agricultural decisions under uncertainty can be better evaluated, purchasing decisions can be optimized, and cargo loading can be planned for the greatest efficiency. Those who are concerned with marketing issues can evaluate media mixes to maximize exposure, plan product introductions, arrange portfolios of assets, assess pricing strategies, and propose optimal sales allocations.

A subset of OR applications is concerned with the location of facilities. Examples include the location of warehouses or retail stores, the layout of workstations within a factory, and the placement of public services such as fire or police stations. Location problems will often contain a function of distance, which may be measured in a variety of ways. Some of the most common location problem structures include median problems, center problems, dispersion problems, covering problems, and layout problems. A smaller subset of location applications concerns problems that occur on networks or in systems that can be represented as networks. Such problems take advantage of network topology and the associated graph theoretic concepts of connectivity and adjacency. An enormous number of decisions related to transportation and physical distribution can be constructed and evaluated using these types of models. Other applications on networks involve things such as pipeline construction, highway patrol scheduling, and school bus routing.

Operations Research as a Decision Tool in Social Science

Although OR has proved its worth for practical applications in military, industrial, management, or commercial interests, there are many other applications for which the motivation is to understand social systems. Because many social systems are extraordinarily complex, the model structures and solution procedures designed in OR are capable of providing insights where other methods would be overwhelmed.

Some of the applications of OR in the social sciences are economic in nature, including research into labor costs and market demand under different conditions, capital budgeting and expansion for the public sector, and budget allocation. Other applications concern the distribution of limited common public resources. Others may give insight into the patterns of criminal activity through "hot-spot" analysis. Political campaign strategies and policy platforms can be designed or analyzed using OR methods. Appropriate candidates for committee assignments or the selection of the most diverse set of applicants for acceptance to a graduate program can be modeled with the OR process. Ecological applications exist when the goal is to minimize the risk of natural hazards, optimize forest management, encourage environmental protection, select sites for natural reserves, or implement pollution controls. Still more applications are designed to increase the efficiency of public services, such as minimizing the response time for emergency personnel, or reduce inequities in access to services or workloads among public servants.

Generally speaking, OR offers a structure for modeling the complex relationships among humans or between humans and the environment. Even though these social systems can be highly complex, OR allows simplified versions of these systems to be modeled in such a way that their individual constraints and variables can be examined and used to generate alternative solutions.

Prospects and Opportunities

Although OR has matured quickly over the decades since its inception, its rapid growth and dissemination into a wide variety of applications areas have opened up new areas for fundamental research. Due to the complex nature and computational complexity of the systems being modeled in OR, there is an ongoing search for new methods (or modifications of existing methods) that will allow a greater number of problem instances to be solved optimally. Investigations into "integer-friendly" formulations—that is, formulations that will generate integer solutions without explicit integrality constraints—are one area of interest. New and variant heuristic solution procedures are developed on a regular basis. Of course, the search for an algorithmic solution to NP-complete OR problems or to special cases of such problems is of constant concern. Although the notions of optimality and fuzzy modeling may seem to be at odds with one another, the notion of parameters that are dynamic has produced substantial interest. Perhaps most importantly, each incremental step in OR allows for a greater understanding of complex systems—often social systems—and the models chosen for these systems expose both our understanding of them and the limitations of our ability to capture and study them.

See Also the Following Article

Heuristics

Further Reading

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